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THE BALLISTIC PROBLEM

BY FRANK GILMAN

THE problem of the motion of a projectile in a medium resisting as any power of the velocity was first solved by John Bernouilli in 1718, but he expressed the coordinates of any point of the trajectory in a form where a final integration is required to determine them numerically; and, as far as the general problem is concerned, no later mathematician has improved on his results.

Modern experiments in gunnery, notably those conducted for the British government (1865–1880) by General Francis Bashforth, have established the fact that for moderate velocities, *i. e.* from 100 to 900 feet a second, the resistance of the air to the passage of the projectile is very nearly proportional to the square of the velocity; that for greater velocities *i. e.* from 900 to 1300 feet a second it is nearly proportional to the cube of the velocity; and that for high velocities, *i. e.* from 1300 to 2780 feet a second it is again very closely proportional to the square of the velocity.

In 1873 General Bashforth published a report of his experiments accompanied with the mathematical theory for the case where the resistance of the air is as the cube of the velocity, and an elaborate set of tables based on the method of quadratures for the integration of the final equations.

In 1890 he published a revised report in which the more difficult case where the resistance is as the square of the velocity was treated in like manner.

The same subject has been somewhat differently treated by Niven,* Siacci,† and Ingalls.‡

In this paper I shall attempt to show that by means of the application of the method of Least Squares to the development of functions the differential equations for the trajectory when the resistance is as the square of the velocity can be integrated without excessive labor, so that they will give results of very considerable exactness.

I shall first consider a method oi very great accuracy but requiring much labor in its application, and I shall afterwards develop much simpler working methods of sufficient accuracy for practical purposes.

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* Niven, *Proceedings of the Royal Society*, London, Vol. 26, 1877.

† Siacci, *Balistique Extérieure;* Traduction annotée par P. Laurent; Paris, 1892.

‡ Ingalls, *Exterior Ballistics*, Washington, 1900.

The following* are the well known differential equations of motion for the case under consideration:

$$cdx = \frac{dp}{p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) - C}, \tag{1}$$

$$cdy = \frac{pdp}{p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) - C}, \tag{2}$$

$$\sqrt{cg}.dt = \frac{-dp}{\sqrt{C - p\sqrt{1+p^2} - \log(p + \sqrt{1+p^2})}}, \tag{3}$$

in which dx denotes the differential of the abscissa and dy the differential of the ordinate, while p, or $\frac{dy}{dx}$, is the tangent of the angle at which the trajectory is inclined to the horizontal.

We have also

$$C = \frac{1}{2ch\cos^2\phi_0} + p_0\sqrt{1+p_0^2} + \log(p_0 + \sqrt{1+p_0^2})$$

in which $c = \frac{g}{k^2}$, g being the acceleration of gravity and k the velocity of the projectile when the resistance becomes equal to its weight; $h = \frac{v_0^2}{2g}$ where v_0 is the initial velocity; ϕ_0 is the initial angle of elevation and p_0 the initial value of p. It is easily seen that C is positive and always greater than

$$p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}.)$$

First method. Let us replace the transcendental function $p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2})$ which we will represent by $f(p)$ by the rational expression $a'p + b'p^3 + c'p^5$ in which the coefficients a', b', and c', are to be calculated by the method of Least Squares, the general form of the equation of condition being as follows:

$$a'p + b'p^3 + c'p^5 - f(p) = 0.$$

Make successively in this equation $p = 0.1$, $p = 0.2$, etc. up to $p = 1.0$, thus forming 10 equations of condition. Forming the normal equations and solving them we obtain

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* v. Bartlett's *Analytical Mechanics*.

$$\begin{aligned}\text{Log } a' &= 0.3012124\\ \text{Log } b' &= 9.5126277\\ \text{Log } (-c') &= 8.4899010\end{aligned}$$

A more detailed exposition of this method of developing irrational and transcendental expressions is given in an article that was published in the *Journal of The Franklin Institute* for June, 1894, entitled "Application of the Method of Least Squares to the Development of Functions."

The above values of a', b', and c' when substituted in the rational expression will make it give values for $f(p)$ whose greatest error will hardly exceed one in the fourth place of decimals, while the integrals, cx and cy, are likely to be even more exact.

In formulas developed by the method of Least Squares (especially when the number of conditional equations is large) the sum of the positive errors in a series of results, will be nearly equal to the sum of the negative errors Therefore if the line $Oafd$ represents the course of a given curve with reference to the axes x and y, its location as determined by the method of Least Squares is shown by the curve Ocd; and although these curves may deviate sensibly from each other at certain points, as at a and c, yet the areas included between the lines Oe, ed and each curve will be practically the same. In other words the corresponding definite integrals will be nearly equal.

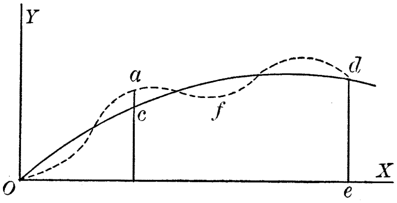

FIG. 1.

The differential equations (1) and (2) may now be written as follows:

$$cc'dx = \frac{dp}{p^5 + \frac{b'}{c'}p^3 + \frac{a'}{c'}p - \frac{C}{c'}}, \tag{4}$$

$$cc'dy = \frac{pdp}{p^5 + \frac{b'}{c'}p^3 + \frac{a'}{c'}p - \frac{C}{c'}} \tag{5}$$

These equations can now be integrated by the method of decomposition into partial fractions.

Let $l = \frac{b'}{c'}$, $m = \frac{a'}{c'}$, $n = -\frac{C}{c'}$ and observe that l and m are negative and n positive.

By Descartes' Rule of Signs $p^5 + lp^3 + mp + n = 0$ has two positive real roots, one negative real root and two imaginary roots; call them α, β, γ, $\delta + \epsilon\sqrt{-1}$, $\delta - \epsilon\sqrt{-1}$. Then

$$\frac{1}{p^5 + lp^3 + mp + n} \equiv \frac{L}{p-\alpha} + \frac{M}{p-\beta} + \frac{N}{p-\gamma} + \frac{Pp+Q}{(p-\delta)^2+\epsilon^2}, \qquad (6)$$

$$L = \left[\frac{p-\alpha}{p^5 + lp^3 + mp + n}\right]_{p=\alpha} = \frac{1}{5\alpha^4 + 3l\alpha^2 + m}.$$

Similarly,

$$M = \frac{1}{5\beta^4 + 3l\beta^2 + m},$$

$$N = \frac{1}{5\gamma^4 + 3l\gamma^2 + m}.$$

Clearing (6) of fractions and writing the coefficients of p^4 and p^3 equal to zero, noting that $\alpha + \beta + \gamma + 2\delta = 0$, we have

$$L + M + N + P = 0, \text{ and } L\alpha + M\beta + N\gamma + 2P\delta + Q = 0,$$

whence

$$P = -(L + M + N),$$

$$Q = -(L\alpha + M\beta + N\gamma + 2P\delta).$$

Equations (4) and (5) now integrate into

$$cc'x = L\log(p-\alpha) + M\log(p-\beta) + N\log(p-\gamma) + \frac{P}{2}\log[(p-\delta)^2+\epsilon^2]$$
$$+ \frac{P\delta + Q}{\epsilon}\tan^{-1}\left(\frac{p-\delta}{\epsilon}\right), \qquad (7)$$

$$cc'y = L\alpha\log(p-\alpha) + M\beta\log(p-\beta) + N\gamma\log(p-\gamma)$$
$$+ \frac{2P\delta + Q}{2}\log[(p-\delta)^2+\epsilon^2] + \frac{\delta(P\delta+Q) - P\epsilon^2}{\epsilon}\tan^{-1}\left(\frac{p-\delta}{\epsilon}\right). \qquad (8)$$

The constants of integration are determined from the condition that when p is equal to its initial value p_0, $x = 0$ and $y = 0$; but the formulas may be used as they stand without the determination of the constants of integration if p is taken between proper limits.

We will now apply these equations in solving the following numerical example which is given by Professor Bashforth in his treatise on the Chronograph, page 97.

Initial angle of elevation $\phi_0 = 30°$; initial velocity $V = 751$ ft. per sec.; diameter of shot $= 6.27$ in.; weight $= 70$lbs. Here $c = 0.0000339822$, $C = 3.455425$.

We first find the five roots of the equation

$$p^5 + lp^3 + mp + n = 0,$$

in which $l = \frac{b'}{c'} = -10.5372354$, $m = \frac{a'}{c'} = -64.7606866$, $n = -\frac{C}{c'} = +111.84085$.

The following are the values of the roots:

$$\alpha = +1.3779718$$
$$\beta = +3.6221343$$
$$\gamma = -4.0293468$$
$$\delta = -\frac{\alpha + \beta + \gamma}{2} = -0.4853796$$
$$\epsilon^2 = 3\delta^2 + l - \gamma(\alpha + \beta) - \alpha\beta = +5.3255100$$

Next computing the five constants by the formulas of the preceding page we get:

$$L = -0.0093670, \quad M = +0.0026236, \quad N = +0.0013514,$$
$$P = +0.0053920, \quad Q = +0.0140848.$$

It is evident that the accuracy of the integrals as given by equations (7) and (8) will depend on the exactness with which the following equation is satisfied:

$$\frac{1}{p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) - C} = \frac{1}{c'}\left[\frac{L}{p-\alpha} + \frac{M}{p-\beta} + \frac{N}{p-\gamma} + \frac{Pp+Q}{(p-\delta)^2+\epsilon^2}\right]. \quad (9)$$

Substituting in this equation the values of the roots and constants already found, we obtain the following table, which gives the values of the two members of this equation corresponding to different values of p:

p	FIRST MEMBER	SECOND MEMBER
0.05	0.29803	0.29803
0.10	0.30721	0.30722
0.15	0.31703	0.31704
0.20	0.32757	0.32758
0.25	0.33895	0.33896
0.30	0.35130	0.35132
0.35	0.36478	0.36479
0.40	0.37957	0.37957
0.45	0.39590	0.39589
0.50	0.41404	0.41403
0.55	0.43436	0.43434
0.60	0.45727	0.45725
0.65	0.48334	0.48331
0.70	0.51327	0.51325
0.75	0.54801	0.54801

In accordance with what has been said in connection with figure 1, it follows that the integrals are likely to be even more exact than might be inferred from the indications of the above table.

Substituting the values of the roots and constants in equation (7), and taking it between the limits $p = \tan 30°$ and $p = 0$, we find $x = 6026.8$ feet, which is the horizontal distance from the origin to where the projectile attains its greatest elevation.

Professor Bashforth finds by quadratures $x = 6032$ ft., which is 5 feet in error.

Taking equation (8) between the same limits, we obtain $y = 1864.4$ ft. Professor Bashforth finds $y = 1867$ ft. In order to find the range we will assume with Bashforth that the angle of fall $= -(37° 3')$, and taking equations (7) and (8) between the limits $p = \tan 30°$ and $p = \tan -(37° 3')$, we obtain

$$x = 11326.1 \text{ ft.} = \text{range},$$
$$y = -\ 7.2 \text{ ft.}$$

According to Bashforth, $x = 11334$ ft., and $y = -6.5$ ft., showing an error of about 8 feet in the range.

The equations (7) and (9) undoubtedly give the coordinates of the trajectory correct to the nearest foot; but their use involves great labor, and they are chiefly valuable as a criterion by which to compare the results of simpler methods.

Second method. If instead of replacing our $f(p) \equiv p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2})$ by $a'p + b'p^3 + c'p^5$ we assume $f(p) = ap + bp^2$, a and b being constants to be determined as before, our method while open to the objection that two separate sets of formulas are required one for the ascending and the other for the descending portion of the trajectory, is very much less laborious in practice than the one we have been considering.

Since $f(-p) = -f(p)$, if we determine a and b on the assumption that p is positive we must use $ap - bp^2$ for $f(p)$ when p is negative.

Integrating the differential equations (1), (2), and (3) on the assumption that $f(p) = ap + bp^2$ we get for the ascending portion of the trajectory:

$$x = \frac{1}{c\sqrt{a^2 + 4bC}} \log \frac{a + 2bp - \sqrt{a^2 + 4bC}}{a + 2bp + \sqrt{a^2 + 4bC}}, \tag{10}$$

$$y = \frac{1}{2bc} \log(C - ap - bp^2) - \frac{ax}{2b}, \tag{11}$$

$$t = \frac{1}{\sqrt{bcg}} \sin^{-1}\left(\frac{a + 2bp}{\sqrt{a^2 + 4bC}}\right). \tag{12}$$

For the descending portion where $f(p) = ap - bp^2$ we get:

$$x = \frac{1}{c\sqrt{a^2 - 4bC}} \log \frac{a - 2bp - \sqrt{a^2 - 4bC}}{a - 2bp + \sqrt{a^2 - 4bC}}, \text{ if } a^2 > 4bC, \tag{13}$$

$$x = \frac{2}{c\sqrt{4bC - a^2}} \tan^{-1}\left(\frac{a - 2bp}{\sqrt{4bC - a^2}}\right), \qquad \text{if } a^2 < 4bC, \tag{13'}$$

$$y = -\frac{1}{2bc} \log(C - ap + bp^2) + \frac{ax}{2b}, \tag{14}$$

$$t = \frac{1}{\sqrt{bcg}} \log\left[\sqrt{C - ap + bp^2} - p\sqrt{b} + \frac{a}{2\sqrt{b}}\right]. \tag{15}$$

As the constants of integration have not been introduced in the above formulas, we must remember in using them to take p between the limits for which x, y and t are to be calculated.

The following table gives the values of a and b corresponding to different angles of elevation.

p	ϕ	log a	log b	a	a^2
0.1	5° 43′	0.3007161	8.6654758	1.998555	3.994222
0.2	11 19	0.2998279	8.9559684	1.994472	3.977917
0.3	16 42	0.2983910	9.1254675	1.987884	3.951682
0.4	21 48	0.2964379	9.2440980	1.978964	3.916298
0. 5	26 34	0.2940089	9.3342507	1.967927	3.872735
0.6	30 58	0.2911508	9.4060248	1.955018	3.822097
0.7	35 0	0.2879118	9.4649223	1.940492	3.765508
0.8	38 40	0.2843400	9.5142827	1.924598	3.704077
0.9	41 59	0.2804821	9.5563036	1.907577	3.638850
1.0	45 0	0.2763813	9.5925171	1.889650	3.570776

In using this table no interpolations will be necessary, as either one of two consecutive sets of values of the a and b functions can be used without making any material difference in the results. The quantities a and b in the table are determined in the usual manner by the method of Least Squares, the general form of the equation of condition being

$$ap + bp^2 - f(p) = 0.$$

Make successively in this equation $p = 0.01$, $p = 0.02$, etc., up to $p = 0.10$, thus forming 10 equations of condition. Solving the resulting normal equations, we obtain the log a and log b given in the first horizontal line of the table. In the same manner the log a and log b found in the second horizontal line of the table are formed from 20 equations of condition, and those in the last horizontal line are formed from 100 equations of condition.

We will now solve by the above equations and table the numerical example that we have already worked by the first method, and in which $C = 3.455425$, $c = 0.0000339822$. Since in this example the initial angle of elevation equals 30°, we take the values of a and b from that line in the table corresponding to $p = 0.6$. Substituting these quantities together with the values of C and c in equations (10) (11) and (12) and taking them between the limits corresponding to $p = \tan 30°$ and $p = 0$, we obtain

$$x = 6026.7 \text{ ft.},$$

$$y = 1864.8 \text{ ft.},$$

$$t = 10.38 \text{ sec.}$$

As the angle of fall is 37° 3′, we take for the descending branch the values of a and b corresponding to $p = 0.8$. Substituting these in the equations (13′) (14) and (15) and taking them between the limits $p = \tan (37° 3')$ and $p = 0$, we find

$$x = 5300.7 \text{ ft.},$$

$$y = -1871.1 \text{ ft.},$$

$$t = 11.13 \text{ sec.}$$

Range $= 6026.7 + 5300.7 = 11327.4$ ft.

Time of flight $= 10.38 + 11.13 = 21.51$ sec.

If we use the values of a and b corresponding to $p = 0.8$ for the ascending branch of the trajectory, we obtain the following for the x, y and t of this branch:

$$x = 6026.2 \text{ ft.},$$

$$y = 1865.3 \text{ ft.},$$

$$t = 10.38 \text{ sec.}$$

From this it appears that we may generally use the same values of a and b for the ascending as for the descending branch, those quantities being taken which correspond to the angle of fall.

We will illustrate by one more example taken from Capt. Ingalls' *Treatise on Exterior Ballistics*, page 104.

A 12-inch service projectile is fired at an angle of departure of 10°, with an initial velocity of 1886 ft. per sec.; it is required to find x_1, y_1, t_1, corresponding to $p = 0$, and x_2, y_2, t_2 for the point where the angle of fall is $-13°$, both sets of coordinates being referred to the mouth of the gun.

In this example $C = 0.72037$, $c = 0.0000254985$. Take a and b from that line of the table corresponding $p = 0.2$. Substituting in equations (10), (11) and (12), and taking p between the limits tan 10° and 0, we find

$$x_1 = 13203.8 \text{ ft.},$$

$$y_1 = 1294.4 \text{ ft.},$$

$$t_1 = 8.46 \text{ sec.}$$

For the descending branch take the a and b corresponding to $p = 0.3$, and substitute in equations (13) (14) and (15), taking p between the limits tan (– 13°) and 0. We thus obtain the following values of the coordinates referred to the vertex of the trajectory:

$$
\begin{aligned}
x &= 9705.3 \text{ ft.},\\
y &= -1027.3 \text{ ft.},\\
t &= 8{\cdot}32 \text{ sec.}
\end{aligned}
$$

Whence

$$
\begin{aligned}
x_2 &= 13203.8 + 9705.3 = 22909.1 \text{ ft.},\\
y_2 &= 1294.4 - 1027.3 = 267.1 \text{ ft.},\\
t_2 &= 8.46 + 8.32 = 16.78 \text{ sec.}
\end{aligned}
$$

Capt. Ingalls working by "Niven's Method" obtains the following values for these quantities:

$$
\begin{aligned}
x_1 &= 13180.7 \text{ ft.},\\
y_1 &= 1291.8 \text{ ft.},\\
x_2 &= 23007.0 \text{ ft.},\\
y_2 &= 241.7 \text{ ft.},\\
t_1 &= 8.44 \text{ sec.},\\
t_2 &= 16.83 \text{ sec.}
\end{aligned}
$$

If the actual law of the resistance is that of the square of the velocity these results of Capt. Ingalls are somewhat erroneous, the x_2 being 98 feet in error.

The more exact equations (7) and (8) give the following values for these coordinates:

$$
\begin{aligned}
x_1 &= 13205.6 \text{ ft.},\\
y_1 &= 1295.5 \text{ ft.},\\
x_2 &= 22908.9 \text{ ft.},\\
y_2 &= 267.3 \text{ ft.}
\end{aligned}
$$

Third method. A rougher method which in point of simplicity leaves little to be desired, and yet is surprisingly accurate in application is based on the assumption that $f(p) \equiv p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) = ap$.

Equations (1), (2) and (3) integrate at once into

$$cx = \frac{1}{a}\log(C - ap). \tag{16}$$

$$cy = \frac{1}{a}\left[p + \frac{C}{a}\log(C - ap)\right], \tag{17}$$

$$\sqrt{cg}\cdot t = \frac{2}{a}\sqrt{C - ap}, \tag{18}$$

and the ascending and descending portions of the trajectory do not need to be dealt with separately.

Here again as the constants of integration are not written we must in using the equations take the difference between the values corresponding to the extreme values of p.

The value of a is to be determined by the method of Least Squares, our equation of condition being $ap - f(p) = 0$; but as we have but one unknown we have merely to compute the arithmetical mean of the values of $\frac{f(p)}{p}$.

The following table gives the values of a for different angles of elevation:

p	ϕ	a	p	ϕ	a
0.05	2° 52′	2.00028	0.55	28 49	2.03364
0.10	5 43	2.00124	0.60	30 58	2.03976
0.15	8 32	2.00272	0.65	33 1	2.04635
0.20	11 19	2.00474	0.70	35 0	2.05339
0.25	14 2	2.00731	0.75	36 52	2.06087
0.30	16 42	2.01041	0.80	38 40	2.06879
0.35	19 17	2.01403	0.85	40 22	2.07711
0.40	21 48	2.01818	0.90	41 59	2.08585
0.45	24 14	2.02284	0.95	43 32	2.09497
0.50	26 34	2.02799	1.00	45 0	2.10447

Applying this third method to the examples already worked by the first method and the second method, we get for the range and time in the first example

$$x = 11335 \text{ ft.},$$
$$t = 21.51 \text{ sec.};$$

and for the coordinates and the time in the second example

$$x_2 = 22906 \text{ ft.},$$
$$y_2 = 266 \text{ ft.},$$
$$t_2 = 16.78 \text{ sec.}$$

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